The nonlinear Schrödinger equation on metric graphs Brown-Yale PDE seminar

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Foreword	Metric graphs	NLS	Ground states	Another notion of ground state?	Take-home message □
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First of all, let me thank:

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you!



A metric graph is made of vertices





A metric graph is made of vertices and of edges joining the vertices or going to infinity.





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- *metric* graphs: the lengths of edges are important.
- the edges going to infinity are halflines and have *infinite length*.
- a metric graph is *compact* if and only if it has a finite number of edges of finite length.

Constructions based on halflines

The halfline

Constructions based on halflines



Constructions based on halflines



Constructions based on halflines



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Periodic graphs





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Infinite trees



Figure: Infinite trees



A metric graph G with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3)



A metric graph \mathcal{G} with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$





A metric graph \mathcal{G} with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions.





A metric graph \mathcal{G} with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions.

$$\int_{\mathcal{G}} f \, \mathrm{d}x := \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x$$

Why studying metric graphs? Physical motivations

Modeling structures where only one spatial direction is important.



A "fat graph" and the underlying metric graph

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on each edge } e \text{ of } \mathcal{G}, \\ \\ & \text{(NLS}_{\mathcal{G}}) \end{cases}$$

$$\begin{aligned} & (-u'' + \lambda u = |u|^{p-2}u & \text{ on each edge } e \text{ of } \mathcal{G}, \\ & u \text{ is continuous} & \text{ for every vertex } v \text{ of } \mathcal{G}, \end{aligned}$$
 (NLS_G)

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(NLS_G)

Given constants p > 2 and $\lambda > 0$, we are interested in solutions $u \in L^2(\mathcal{G})$ of the differential system

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(NLS_G)

The condition on the sum of derivatives is called *Kirchhoff's condition*.

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Kirchhoff's condition: degree-one nodes





Kirchhoff's condition in general: outgoing derivatives





The soliton φ_{λ} on the real line



The set of nonzero solutions of $(NLS_{\mathcal{G}})$ is

$$\mathcal{S}_\lambda(\mathbb{R}) = \left\{\pm arphi_\lambda(x+\mathsf{a}) \; \Big| \; \mathsf{a} \in \mathbb{R}
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where the soliton φ_{λ} is the unique positive and even solution to

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The nonlinear Schrödinger equation

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The nonlinear Schrödinger equation The half-line: $\mathcal{G} = \mathbb{R}^+$



$$\mathcal{S}_{\lambda}(\mathbb{R}^+) = \left\{\pm \varphi_{\lambda}(x)_{|\mathbb{R}^+}
ight\}$$

Solutions are *half-solitons*: no more translations!
The nonlinear Schrödinger equation

The positive solution on the 3-star graph



The nonlinear Schrödinger equation



The nonlinear Schrödinger equation



The nonlinear Schrödinger equation



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Variational formulation

We work on the Sobolev space

$$H^1(\mathcal{G}) := \Big\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G}) \Big\}.$$

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Solutions of $(NLS_{\mathcal{G}})$ correspond to critical points of the action functional

$$J_{\lambda}(u) := rac{1}{2} \|u'\|_{L^2(\mathcal{G})}^2 + rac{\lambda}{2} \|u\|_{L^2(\mathcal{G})}^2 - rac{1}{p} \|u\|_{L^p(\mathcal{G})}^p.$$

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The level of the soliton φ_{λ} plays an important role in our analysis:

$$s_{\lambda} := J_{\lambda}(\varphi_{\lambda}).$$

The Euler-Lagrange equation associated to J_{λ}

The differential of $J_{\lambda}: H^1(\mathcal{G}) \to \mathbb{R}$ is given by

$$J'_{\lambda}(u)[v] = \int_{\mathcal{G}} u'(x)v'(x) \,\mathrm{d}x + \lambda \int_{\mathcal{G}} u(x)v(x) \,\mathrm{d}x - \int_{\mathcal{G}} |u(x)|^{p-2}u(x)v(x) \,\mathrm{d}x$$

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If arphi has compact support in the interior of an edge $e={}_{\mathrm{AB}}$, we have

$$0 = J'_{\lambda}(u)[\varphi]$$

= $\int_{e} u'(x)\varphi'(x) dx + \lambda \int_{e} u(x)\varphi(x) dx - \int_{e} |u(x)|^{p-2}u(x)\varphi(x) dx$

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so that $-u'' + \lambda u = |u|^{p-2}u$ on edges of \mathcal{G} .

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Kirchhoff's condition

Let A be a vertex of \mathcal{G} and let B_1, \ldots, B_D be the vertices adjacent to A.

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Let A be a vertex of \mathcal{G} and let B_1, \ldots, B_D be the vertices adjacent to A. Define φ so that it is affine on all edges of \mathcal{G} , $\varphi(A) = 1$ and $\varphi(V) = 0$ for all vertices $V \neq A$. Denote $e_i := AB_i$.



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= $\sum_{1 \le i \le D} \left(\int_{e_i} u' \varphi' \, \mathrm{d}x + \lambda \int_{e_i} u \varphi \, \mathrm{d}x - \int_{e_i} |u|^{p-2} u \varphi \, \mathrm{d}x \right)$

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+ $\sum_{1 \le i \le D} \int_{e_i} \left(\underbrace{-u'' + \lambda u - |u|^{p-2}u}_{=0} \right) \varphi(x) \, \mathrm{d}x$

so that $\sum_{1 \le i \le D} \frac{du}{dx_{e_i}}(A_i) = 0$, which is Kirchhoff's condition.

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The Nehari manifold

The functional J_{λ} is not bounded from below on $H^1(\mathcal{G})$, since if $u \neq 0$ then

$$J_{\lambda}(tu)=rac{t^2}{2}\|u'\|_{L^2(\mathcal{G})}^2+rac{\lambda t^2}{2}\|u\|_{L^2(\mathcal{G})}^2-rac{t^p}{p}\|u\|_{L^p(\mathcal{G})}^p\xrightarrow[t o\infty]{}-\infty.$$

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A common strategy is to introduce the Nehari manifold $\mathcal{N}_{\lambda}(\mathcal{G})$, defined by

$$\begin{split} \mathcal{N}_{\lambda}(\mathcal{G}) &:= \Big\{ u \in H^1(\mathcal{G}) \setminus \{0\} \mid J_{\lambda}'(u)[u] = 0 \Big\} \\ &= \Big\{ u \in H^1(\mathcal{G}) \setminus \{0\} \mid \|u'\|_{L^2(\mathcal{G})}^2 + \lambda \|u\|_{L^2(\mathcal{G})}^2 = \|u\|_{L^p(\mathcal{G})}^p \Big\}. \end{split}$$

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If $u \in \mathcal{N}_{\lambda}(\mathcal{G})$, then

$$J_{\lambda}(u) = \Big(rac{1}{2} - rac{1}{p}\Big) \|u\|_{L^p(\mathcal{G})}^p.$$

In particular, J_{λ} is bounded from below on $\mathcal{N}_{\lambda}(\mathcal{G})$.

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Action ground states

• "Ground state" action level:

$$\mathcal{J}_\mathcal{G}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u)$$



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$$\mathcal{J}_\mathcal{G}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u)$$

Ground state: function u ∈ N_λ(G) with level J_G(λ). If it exists, it is a solution of the differential system (NLS_G).

A very useful tool: cutting solitons on halflines

Proposition

Assume that \mathcal{G} has at least one halfline. Then,

 $\mathcal{J}_{\mathcal{G}}(\lambda) \leq s_{\lambda} := J_{\lambda}(\varphi_{\lambda})$

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An existence Theorem

Theorem (Adami-Serra-Tilli 2015, Dovetta-De Coster-G.-Serra-Troestler 2024)

Let G be a metric graph with finitely many edges, including at least one halfline. Let p > 2 and $\lambda > 0$ be real numbers. If,

$$\mathcal{J}_{\mathcal{G}}(\lambda) < s_{\lambda}$$

then action ground states exist.

Some graphs which admit action ground states



Figure: Examples of graphs admitting action ground states. (a): the \mathcal{T} -graph; (b): the signpost; (c): the tadpole; (d): the 3-fork.

Decreasing rearrangement on the halfline



For all $1 \leq p \leq +\infty$,

 $||u||_{L^{p}(\mathcal{G})} = ||u^{*}||_{L^{p}(0,|\mathcal{G}|)}.$
Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Then its decreasing rearrangement u^* belongs to $H^1(0, |\mathcal{G}|)$, and one has

 $\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \le \|u'\|_{L^2(\mathcal{G})}.$

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Friedlander, L. *Extremal properties of eigenvalues for a metric graph.* Ann. Inst. Fourier (Grenoble) **55** (2005) no. 1, 199–211. Foreword Metric graphs NLS Ground states Another notion of ground state? Take-home message

The Pólya–Szegő inequality A simple case: affine functions

We assume that u is piecewise affine.



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We consider a small open interval $I \subseteq u(\mathcal{G})$ so that $u^{-1}(I)$ consists of a disjoint union of open intervals on which u is affine.

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The nonlinear Schrödinger equation on metric graphs

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Original contribution to $||u'||_{L^2}^2$:

$$\mathsf{A} := \ell_1 \frac{|\mathcal{I}|^2}{\ell_1^2} + \ell_2 \frac{|\mathcal{I}|^2}{\ell_2^2} + \ell_3 \frac{|\mathcal{I}|^2}{\ell_3^2} + \ell_4 \frac{|\mathcal{I}|^2}{\ell_4^2}$$

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Metric graphs NLS Ground states Another notion of ground state? Take-home message

The Pólya–Szegő inequality A simple case: affine functions

Original contribution to $||u'||_{L^2}^2$:

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Foreword

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Foreword

Proposition

Let G be a metric graph with finitely many edges, including at least one halfline. Let p > 2 and $\lambda > 0$ be real numbers. Then,

$$\mathcal{J}_\mathcal{G}(\lambda) \geq rac{1}{2} J_\lambda(arphi_\lambda).$$

Proof.

One may assume that $u \ge 0$.

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$$\begin{split} \|u^*\|_{L^2(0,+\infty)} &= \|u\|_{L^2(\mathcal{G})}, \\ \|u^*\|_{L^p(0,+\infty)} &= \|u\|_{L^p(\mathcal{G})}, \\ \|(u^*)'\|_{L^2(0,+\infty)} \leq \|u'\|_{L^2(\mathcal{G})}. \end{split}$$

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Then, one shows that for a suitable t > 0, the function tu^* belongs to $\mathcal{N}_{\lambda}(0, +\infty)$ and is such that

$$J_{\lambda,\mathcal{G}}(u) \geq J_{\lambda,[0,+\infty[}(tu^*).$$

 Foreword
 Metric graphs
 NLS
 Ground states
 Another notion of ground state?
 Take-home message

A refined Pólya–Szegő inequality...

... or the importance of the number of preimages

Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Let $N \ge 1$ be an integer. Assume that, for almost every $t \in]0, ||u||_{\infty}[$, one has

$$u^{-1}({t}) = {x \in \mathcal{G} \mid u(x) = t} \ge N.$$

Then one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \frac{1}{N} \|u'\|_{L^2(\mathcal{G})}.$$



Definition (Adami, Serra, Tilli (Calc. Var. PDEs. 2014))

We say that a metric graph \mathcal{G} satisfies assumption (H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty[\rightarrow \mathcal{G} \text{ parameterized} by arclength, with disjoint images except for an at most countable number of points, and such that <math>\gamma_1(0) = \gamma_2(0) = x_0$.



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Consequence: all nonnegative $H^1(\mathcal{G})$ functions have at least two preimages for almost every $t \in]0, ||u||_{\infty}[$.

Non-existence of ground states

Theorem (Adami, Serra, Tilli (Calc. Var. PDEs. 2014))

If a metric graph \mathcal{G} satisfies assumption (H), then

 $\mathcal{J}_{\mathcal{G}}(\lambda) = s_{\lambda}$

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Non-existence of ground states

Theorem (Adami, Serra, Tilli (Calc. Var. PDEs. 2014))

If a metric graph \mathcal{G} satisfies assumption (H), then

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but it is never achieved, unless G is isometric to one of the exceptional graphs depicted in the next two slides.

Foreword Metric graphs NLS Ground states Another notion of ground state? Take-home message

Non-existence of ground states Exceptional graphs: the real line



Foreword Metric graphs NLS Ground states Another notion of ground state? Take-home message

Non-existence of ground states Exceptional graphs: the real line with a tower of circles





Another action level

■ Minimal level attained by the solutions of (NLS_G):

$$\sigma_{\lambda}(\mathcal{G}) := \inf_{u \in \mathcal{S}_{\mathcal{G}}(\lambda)} J_{\lambda}(u).$$



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 Minimal action solution: solution u ∈ S_G(λ) of the differential system (NLS_G) of level σ_λ(G).

Foreword Me	etric graphs	NLS	Ground states	Another notion of ground state?	Take-home message □
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Foreword	Metric graphs	NLS	Ground states	Another notion of ground state?	Take-home message □

An analysis shows that four cases are possible:

A1) $\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\lambda}(\mathcal{G})$ and both infima are attained;

Foreword	Metric graphs	NLS	Ground states	Another notion of ground state?	Take-home message □

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Theorem (De Coster, Dovetta, G., Serra (Calc. Var. PDEs. 2023))

For every p > 2, every $\lambda > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph G where this alternative occurs.
Foreword Metric graphs	NLS	Ground states	Another notion of ground state?	Take-home message □
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Foreword Met	ric graphs NLS	Ground states	Another notion of ground state?	Take-home message □
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N-star graphs, $N \ge 3$

$$s_{\lambda} = \mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_{\lambda}(\mathcal{G}) = \frac{N}{2}s_{\lambda}$$



Case A2 $\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\lambda}(\mathcal{G})$ and neither infima is attained





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Why studying metric graphs? Mathematical motivations

Main message

Metric graphs allow to study interesting *one dimensional* problems and are much richer than the usual class of intervals of \mathbb{R} .

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Replacing \mathcal{G} by noncompact smooth open sets $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)$, one expects that the four cases A1, A2, B1, B2 actually occur.

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Replacing \mathcal{G} by noncompact smooth open sets $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)$, one expects that the four cases A1, A2, B1, B2 actually occur. However, to this day, *it remains on open problem*!

Thank	s!

Atomtronics

Cases A2 and B2: what's going on?

Thanks for your attention!



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Thanks!	References	Atomtronics	Cases A2 and B2: what's going on?

- Adami R., Serra E., Tilli P., *NLS ground states on graphs*, Calc. Var. 54, 743–761 (2015).
- Adami, R., Serra, E., Tilli, P. (2015). Lack of Ground State for NLSE on Bridge-Type Graphs. In: Mugnolo, D. (eds) Mathematical Technology of Networks. Springer Proceedings in Mathematics & Statistics, vol 128. Springer, Cham. https://doi.org/10.1007/978-3-319-16619-3 1

Thanks!	References	Atomtronics	Cases A2 and B2: what's going on?

- De Coster C., Dovetta S., Galant D., Serra E. On the notion of ground state for nonlinear Schrödinger equations on metric graphs. Calc. Var. 62, 159 (2023).
- De Coster C., Dovetta S., Galant D., Serra E., Troestler C., *Constant sign and sign changing NLS ground states on noncompact metric graphs*. ArXiV preprint: https://arxiv.org/abs/2306.12121.

Overviews of the subject

- Adami R. Ground states of the Nonlinear Schrodinger Equation on Graphs: an overview (Lisbon WADE). https://www.youtube.com/watch?v=G-FcnRVvoos (2020)
- Riccardo Adami, Enrico Serra, and Paolo Tilli.
 Nonlinear dynamics on branched structures and networks.
 Riv. Math. Univ. Parma (N.S.), 8(1):109–159, 2017.
- Kairzhan A., Noja D., Pelinovsky D. *Standing waves on quantum graphs.* J. Phys. A: Math. Theor. 55 243001 (2022)

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- This is really remarkable: *macroscopic quantum phenomenon!*
- Since 2000: emergence of *atomtronics*, which studies circuits guiding the propagation of ultracold atoms.

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One obtains

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SO

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Damien Galant

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The loops \mathcal{L}_i have length N and \mathcal{B} is made of N edges of length 1.

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A second, periodic, graph



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What's going on in case B2? Two problems at infinity

Since \mathcal{G}_N and $\widetilde{\mathcal{G}}_N$ satisfy (H) and contain halflines, one has

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but that $\sigma_{\lambda}(\mathcal{G}_N)$ is not attained.

■ Therefore, for large *N*, we have that

$$s_{\lambda} = c_{\lambda}(\mathcal{G}_N) < \sigma_{\lambda}(\mathcal{G}_N),$$

and neither infima is attained, as claimed.